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Existence and multiplicity of solutions for a class of quasilinear elliptic equations: An Orlicz–Sobolev space setting ☆

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ABSTRACT

In this paper, we study the existence and multiplicity of solutions of the following quasilinear elliptic problem

$$\begin{cases} -\mathrm{div}\big(a(|\nabla u|)\nabla u\big) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$. The existence and multiplicity of solutions are obtained by genus theory, symmetric mountain pass lemma, fountain theorem and dual fountain theorem, respectively.

1. Introduction

In this paper, we discuss the existence and multiplicity of solutions of the following boundary value problem

$$(\mathrm{P}) \quad \begin{cases} -\mathrm{div}\big(a(|\nabla u|)\nabla u\big) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \in R^N$ ia a bounded domain with smooth boundary $\partial\Omega$. The function $p(t) := ta(|t|)$ is an increasing homeomorphism from R onto itself. Especially, when $a(t) = |t|^{p-2}$, problem (P) is the well-known p-Laplacian equation. There is a large number of papers on the existence of solutions for p-Laplacian equation. We refer to [7,12–15] and the references therein for some results. In this paper, we study problem (P) in an Orlicz–Sobolev space. There are several papers to consider problem (P) in the Orlicz–Sobolev spaces, see [6,9–11].

Problem (P) possesses more complicated nonlinearities, for example, it is inhomogeneous, so in the discussions, some special techniques will be needed. The inhomogeneous nonlinearities have important physical background, e.g.,

(a) nonlinear elasticity: $P(t) = (1+t^2)^\gamma - 1$, $\gamma > \frac{1}{2}$,
(b) plasticity: $P(t) = t^\alpha(\log(1+t))^\beta$, $\alpha \geqslant 1$, $\beta > 0$,
(c) generalized Newtonian fluids: $P(t) = \int_0^t s^{1-\alpha}(\sinh^{-1} s)^\beta\, ds$, $0 \leqslant \alpha \leqslant 1$, $\beta > 0$.

The paper is organized as follows. In Section 2, we present some preliminary knowledge on the Orlicz–Sobolev spaces . In Section 3, we give the main results and their proofs.

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2. Preliminaries

The function a is such that $p : R \to R$ defined by

$$p(t) = \begin{cases} a(|t|)t, & t \neq 0, \\ 0, & t = 0, \end{cases} \tag{2.1}$$

is an increasing homeomorphism from R onto itself and the continuous function $f(x,t) \in C(\bar{\Omega} \times R, R)$ satisfies $f(x,0) = 0$, $x \in \bar{\Omega}$.

Obviously, problem (P) allows a nonhomogeneous function p in the differential operator defining problem (P). To deal with this situation we introduce an Orlicz–Sobolev space setting for problem (P) as follows:

Let

$$P(t) := \int_0^t p(s)\,ds, \qquad \widetilde{P}(t) := \int_0^t p^{-1}(s)\,ds, \quad t \in R, \tag{2.2}$$

then P and $\widetilde{P}$ are complementary N-functions (see [1,16,17]), which define the Orlicz spaces $L^P := L^P(\Omega)$ and $L^{\widetilde{P}} := L^{\widetilde{P}}(\Omega)$, respectively.

Throughout this paper, we assume the following condition on P:

(p) $1 < p^- := \inf_{t>0} \frac{tp(t)}{P(t)} \leqslant p^+ := \sup_{t>0} \frac{tp(t)}{P(t)} < +\infty$.

Under the condition (p), the Orlicz space L^P coincides with the set (equivalence classes) of measurable functions $u : \Omega \to R$ such that

$$\int_\Omega P\big(|u|\big)\,dx < +\infty \tag{2.3}$$

and is equipped with the (Luxemburg) norm, i.e.

$$|u|_P := \inf\left\{k > 0,\ \int_\Omega P\left(\frac{|u|}{k}\right)dx < 1\right\}.$$

We will denote by $W^{1,P}(\Omega)$ the corresponding Orlicz–Sobolev space with the norm

$$\|u\|_{W^{1,P}(\Omega)} := |u|_P + \|\nabla u\|_P$$

and define $W_0^{1,P}(\Omega)$ as the closure of C_0^∞ in $W^{1,P}(\Omega)$.

Let us now introduce the Orlicz–Sobolev conjugate P_* of P, which is given by

$$P_*^{-1}(t) := \int_0^t \frac{P^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau \tag{2.4}$$

(see [1,17]), where we suppose that

$$\lim_{t\to 0}\int_t^1 \frac{P^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau < +\infty \quad \text{and} \quad \lim_{t\to\infty}\int_1^t \frac{P^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau = +\infty. \tag{2.5}$$

In the case $P(t) = \frac{|t|^p}{p}$, (2.5) holds if and only if $N > p$.

Let

$$p_*^- := \inf_{t>0} \frac{tP_*'(t)}{P_*(t)}, \qquad p_*^+ := \sup_{t>0} \frac{tP_*'(t)}{P_*(t)}. \tag{2.6}$$

Throughout this paper, we assume that p^+ and p_*^- satisfy the following condition:

(p_*) $p^+ < p_*^-$.

We will make the following assumptions on $f(x,t)$:

(f_*) There exists an odd increasing homeomorphism h from R to R, and nonnegative constants a_1, a_2 such that

$$\bigl|f(x,t)\bigr| \leqslant a_1 + a_2 h\bigl(|t|\bigr), \quad \forall t \in R,\ \forall x \in \bar{\Omega}$$

and

$$\lim_{t\to+\infty} \frac{H(t)}{P_*(kt)} = 0, \quad \forall k > 0,$$

where

$$H(t) := \int_0^t h(s)\,ds. \tag{2.7}$$

Let

$$\widetilde{H}(t) := \int_0^t h^{-1}(s)\,ds, \tag{2.8}$$

then we can obtain complementary N-functions which define corresponding Orlicz spaces L^H and L^{H^*}.

Similar to condition (p), we also assume the following condition on H:

(h) $1 < h^- := \inf_{t>0} \frac{th(t)}{H(t)} \leqslant h^+ := \sup_{t>0} \frac{th(t)}{H(t)} < +\infty$.

In this paper, we will use the following equivalent norm on $W_0^{1,P}(\Omega)$:

$$\|u\| := \inf\biggl\{k > 0\colon \int_\Omega P\biggl(\frac{|\nabla u|}{k}\biggr)dx < 1\biggr\}.$$

In order to prove our results, we now state some useful lemmas.

Lemma 2.1. *(See [1].) Under the condition (p), the spaces $L^P(\Omega)$, $W_0^{1,P}(\Omega)$ and $W^{1,P}(\Omega)$ are separable and reflexive Banach spaces.*

Lemma 2.2. *(See [1].) Under the condition (f_*), the imbedding*

$$W^{1,P}(\Omega) \hookrightarrow L^H(\Omega)$$

is compact.

Lemma 2.3. *(See [9].) Let $\rho(u) = \int_\Omega P(u)\,dx$, we have*

(1) *if $|u|_P < 1$, then $|u|_P^{p^+} \leqslant \rho(u) \leqslant |u|_P^{p^-}$,*
(2) *if $|u|_P > 1$, then $|u|_P^{p^-} \leqslant \rho(u) \leqslant |u|_P^{p^+}$,*
(3) *if $0 < t < 1$, then $t^{p^+} P(u) \leqslant P(tu) \leqslant t^{p^-} P(u)$,*
(4) *if $t > 1$, then $t^{p^-} P(u) \leqslant P(tu) \leqslant t^{p^+} P(u)$.*

Remark 2.1. Since problem (P) possesses inhomogeneous nonlinearities, we utilize Lemma 2.3 to overcome the nonhomogeneous difficulty.

3. Main results and proofs

In this section, we assume that $N \geqslant 1$ and $X = W_0^{1,P}(\Omega)$. $u \in X$ is called a weak solution of problem (P) if

$$\int_\Omega a\bigl(|\nabla u|\bigr)\nabla u \nabla \phi\,dx = \int_\Omega f(x,u)\phi\,dx, \quad \forall \phi \in X. \tag{3.1}$$

Set

$$\mathcal{P}(u)=\int_{\Omega} P\big(|\nabla u|\big)\,dx, \quad \forall u \in X, \tag{3.2}$$

$$F(x,t)=\int_{0}^{t} f(x,s)\,ds, \quad \forall (x,t)\in \bar{\Omega}\times R, \tag{3.3}$$

$$I(u)=\int_{\Omega} P\big(|\nabla u|\big)\,dx-\int_{\Omega} F(x,u)\,dx, \quad \forall u \in X, \tag{3.4}$$

and we know that the critical points of I are just the weak solutions of problem (P) (see [6]).

Let

$$\mathcal{F}(u)=\int_{\Omega} F(x,u)\,dx, \tag{3.5}$$

then $I(u)=\mathcal{P}(u)-\mathcal{F}(u)$.

Lemma 3.1.

(1) *[10,11] The functional $\mathcal{P}\in C^1(X,R)$ is convex and sequentially weakly lower semi-continuous and*

$$\mathcal{P}'(u)\phi=\int_{\Omega} p\big(|\nabla u|\big)\nabla\phi\,dx, \quad \forall u,\phi\in X.$$

Moreover, the mapping $\mathcal{P}': X\to X^*$ is bounded homeomorphism, and is of type (S^+), namely,

$$u_n \rightharpoonup u \quad \text{and} \quad \limsup_{n\to\infty}\mathcal{P}'(u_n)(u_n-u)\leqslant 0 \quad \text{imply that} \quad u_n\to u \quad \text{in } X. \tag{3.6}$$

(2) *[6] The functionals $\mathcal{F}(u)$ and $\mathcal{G}(u): X\to R$ are sequentially weakly continuous, $\mathcal{F}(u)\in C^1(X,R)$, and for all $u,\phi\in X$,*

$$\mathcal{F}'(u)\phi=\int_{\Omega} f(x,u)\phi\,dx.$$

The mapping $\mathcal{F}': X\to X^*$ is weakly–strongly continuous, namely,

$$u_n \rightharpoonup u \quad \text{implies that} \quad \mathcal{F}'(u_n)\to\mathcal{F}'(u), \tag{3.7}$$

where $\rightharpoonup$ and $\to$ denote the weak and strong convergences in X, respectively.

Let X be a separable and reflexive Banach space, then there exist (see [4]) $\{e_n\}_{n=1}^{\infty}\subset X$ and $\{e_n^*\}_{n=1}^{\infty}\subset X^*$ such that

$$e_n^*(e_m)=\delta_{n,m}=\begin{cases}1, & \text{if } n=m,\\ 0, & \text{if } n\neq m,\end{cases}$$
$$X=\overline{\operatorname{span}}\{e_n\colon n=1,2,\ldots\}, \qquad X^*=\overline{\operatorname{span}}\big\{e_n^*\colon n=1,2,\ldots\big\}.$$

For $k=1,2,\ldots,$ denote

$$X_k=\overline{\operatorname{span}}\{e_k\}, \qquad Y_k=\bigoplus_{j=1}^{k}X_j, \qquad Z_k=\overline{\bigoplus_{j=k}^{\infty}X_j}. \tag{3.8}$$

Lemma 3.2. *(See [8].) Assume that $\mathcal{F}(u): X\to R$ is weakly–strongly continuous, $\mathcal{F}(0)=0$ and $\gamma>0$ is a given positive number. Set*

$$\beta_k=\sup_{u\in Z_k,\,\|u\|\leqslant\gamma}\big|\mathcal{F}(u)\big|, \tag{3.9}$$

then $\beta_k\to 0$ as $k\to\infty$.

The main results of this paper are Theorems 3.1–3.3 below. In what follows, the conditions (p), (p_*), (f_*) and (h) are assumed for Theorems 3.1–3.3. For the condition (f_*), the $\lim_{t\to+\infty}\frac{H(t)}{P_*(kt)}=0$ means f is *subcritical*. When $a(t)=1$, Theorems 3.1–3.3 are corresponding to the cases of f being sublinear, superlinear and concave–convex, respectively.

Theorem 3.1. *Assume that f satisfies the condition (f_*) and $h^+ < p^-$, then:*

(1) *Problem* (P) *has a solution.*
(2) *Furthermore, if f has the following properties,*
  (A_1) $f(x,-t) = -f(x,t)$ $\forall (x,t) \in \Omega \times R$,
  (A_2) *there exists a $\delta > 0$ such that*

$$f(x,t) \geqslant ct^{q-1} \quad \text{for } x \in R^N \text{ and } 0 < t \leqslant \delta, \tag{3.10}$$

where c is a constant and $q < p^-$,
then problem (P) *has a sequence of solutions $\{\pm u_k\colon k = 1,2,\ldots\}$ such that $I(\pm u_k) < 0$ and $I(\pm u_k) \to 0$ as $k \to \infty$.*

Proof. (1) Lemma 3.1 implies that $I(u)$ is weakly lower semi-continuous. We need to show that $I(u)$ is coercive on X, i.e., $I(u) \to +\infty$ as $\|u\| \to \infty$.

Using (f_*) and Lemma 2.2, when $\|u\| \geqslant 1$, we have

$$\begin{aligned} \left|\mathcal{F}(u)\right| &\leqslant \int_\Omega \left|F(x,u)\right| dx \leqslant \int_\Omega \left(C_1 + C_2 H(u)\right) dx \\ &\leqslant C_3 + C_4 |u|_H^{h^+} \leqslant C_3 + C_5 \|u\|^{h^+}. \end{aligned} \tag{3.11}$$

So we obtain

$$I(u) \geqslant \|u\|^{p-} - \|u\|^{h^+} - C, \tag{3.12}$$

and $h^+ < p^-$ gives the coercivity of $I(u)$, hence $I(u)$ attains its minimum on X, this provides a solution of problem (P).

(2) Since $I(u)$ is coercive, by Lemma 3.1, it is easy to see that $I(u)$ satisfies (PS) condition. By (A_1), $I(u)$ is an even functional. Denote by $\gamma(A)$ the genus of A (see [5,18]). Set

$$\begin{aligned} &\Sigma = \left\{A \subset X \setminus \{0\}\colon A \text{ is compact and } A = -A\right\}, \\ &\Sigma_k = \left\{A \in \Sigma\colon \gamma(A) \geqslant k\right\}, \quad k = 1,2,\ldots, \\ &c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} I(u), \quad k = 1,2,\ldots. \end{aligned}$$

We have

$$-\infty < c_1 \leqslant c_2 \leqslant \cdots \leqslant c_k \leqslant c_{k+1} \leqslant \cdots.$$

Next we will show that $c_k < 0$ for every k.

Let us choose a k-dimensional linear subspace E_k of X such that $E_k \subset C_0^\infty(\Omega)$. Since all the norms on E_k are equivalent to each other, there exists $\rho_k \in (0,1)$ such that $u \in E_k$ with $\|u\| \leqslant \rho_k$. This implies $|u|_{L^\infty} \leqslant \delta$. Set

$$S_{\rho_k}^{(k)} = \left\{u \in E_k\colon \|u\| = \rho_k\right\}. \tag{3.13}$$

By the compactness of $S_{\rho_k}^{(k)}$ and (A_2), there exists a constant d_k such that

$$\int_\Omega c|u|^q\, dx \geqslant d_k, \quad \forall u \in S_{\rho_k}^{(k)}. \tag{3.14}$$

For $u \in S_{\rho_k}^{(k)}$ and $t \in (0,1)$, we have

$$\begin{aligned} I(tu) &= \int_\Omega P\left(|\nabla tu|\right) dx - \int_\Omega F(x,tu)\, dx \\ &\leqslant t^{p^-} \rho_k^{p^-} - \int_\Omega ct^q |u|^q\, dx \\ &\leqslant t^{p^-} \rho_k^{p^-} - t^q d_k. \end{aligned} \tag{3.15}$$

As $q < p^-$, we can find $t_k \in (0,1)$ and $\varepsilon_k > 0$ such that

$$I(t_k u) \leqslant -\varepsilon_k < 0, \quad \forall u \in S_{\rho_k}^{(k)},$$

that is

$$I(u) \leqslant -\varepsilon_k < 0, \quad \forall u \in S^{(k)}_{t_k\rho_k}.$$

We know that $\gamma(S^{(k)}_{t_k\rho_k}) = k$, so $c_k \leqslant -\varepsilon_k < 0$.

By the genus theory (see [5,18]), each c_k is a critical value of I, hence there is a sequence of solutions $\{\pm u_k\colon k = 1, 2, \ldots\}$ of problem (P) such that $I(\pm u_k) = c_k < 0$.

It remains to prove $c_k \to 0$ as $k \to \infty$.

By the coerciveness of $I(u)$, there exists a constant $\gamma > 0$ such that $I(u) > 0$ when $\|u\| \geqslant \gamma$.

Taking arbitrarily $A \in \Sigma_k$, then $\gamma(A) \geqslant k$. Let Y_k and Z_k be the subspaces of X as mentioned in (3.8), according to the properties of genus we know that $A \cap Z_k \neq \emptyset$. Let

$$\beta_k = \sup_{u \in Z_k,\ \|u\| \leqslant \gamma} \bigl|\mathcal{F}(u)\bigr|, \tag{3.16}$$

by Lemma 3.2 we have $\beta_k \to 0$ as $k \to \infty$. When $u \in Z_k$ and $\|u\| \leqslant \gamma$, we have

$$I(u) = \mathcal{P}(u) - \mathcal{F}(u) \geqslant -\mathcal{F}(u) \geqslant -\beta_k, \tag{3.17}$$

hence

$$\sup_{u \in A} I(u) \geqslant -\beta_k, \tag{3.18}$$

and then $c_k \geqslant -\beta_k$, this implies $c_k \to 0$ as $k \to \infty$. □

Remark 3.1. Theorem 3.1 can also be proved by applying an abstract theorem in [19].

Theorem 3.2. *Assume that f satisfies the conditions (f_*),*

(B_1) *There is a positive constant $\alpha > p^+$ such that*

$$0 < \alpha F(x,t) \leqslant t f(x,t) \quad \forall x \in \overline{\Omega},\ t \neq 0,$$

and

(B_2) *$f(x,t) = o(p(|t|))$ as $|t| \to 0$, uniformly for $x \in \overline{\Omega}$,*

then problem (P) *has a nontrivial solution which corresponds to the positive critical value. If furthermore f satisfies*

(B_3) *$f(x,-t) = -f(x,t)$, $\forall (x,t) \in \overline{\Omega} \times R$,*

then problem (P) *has infinity many pairs of solutions which correspond to the positive critical values.*

Proof. (1) Conclusion (1) has been proved in [6]. For readers convenience we will sketch the proof of the conclusion. In [6], through imposing the appropriate conditions on $p(t)$, Ph. Clément and his coauthors construct an Orlicz–Sobolev space setting for problem (P). Assume that the conditions (p), (p_*) and (f_*) are satisfied, then they show that the embedding $W_0^{1,P}(\Omega) \hookrightarrow L^H(\Omega)$ is compact and the corresponding functional I is C^1 with Fréchet derivative given by

$$\bigl\langle I'(u), v\bigr\rangle = \int_\Omega a\bigl(|\nabla u|\bigr)\nabla u \nabla v\, dx - \int_\Omega f(x,u) v\, dx, \quad \forall u, v \in W_0^{1,P}(\Omega). \tag{3.19}$$

And then they prove that the functional I satisfies (PS) condition and mountain pass geometry conditions. Using the well-known mountain pass theorem (see [6]) they obtain this conclusion.

(2) Conclusion (2) can be reached through the mountain pass lemma of the symmetric form (see [2,20]). □

Theorem 3.3. *Let $f(x,t) = \lambda|u|^{m-2}u + \mu|u|^{n-2}u$, where $1 < m < p^- \leqslant p^+ < n < p_*^-$ and $\lambda, \mu > 0$. Then we have*

(1) *problem* (P) *has solutions $\{\pm u_k\}_{k=1}^\infty$ such that $I(\pm u_k) \to +\infty$ as $k \to +\infty$,*
(2) *problem* (P) *has solutions $\{\pm v_k\}_{k=1}^\infty$ such that $I(\pm v_k) < 0$, $I(\pm v_k) \to +\infty$ as $k \to 0$.*

We will use the following *fountain theorem* and the *dual fountain theorem* to prove Theorem 3.3.

Proposition 3.1 *(Fountain theorem). (See [20].) Under the assumption*

(C_1) *X is a Banach space, $I \in C^1(X,R)$ is an even functional, the subspaces X_k, Y_k and Z_k are defined by* (3.8).

If for each $k = 1, 2, \dots$, *there exists* $\rho_k > r_k > 0$ *such that*

(C_2) $\inf_{u \in Z_k, \|u\|=r_k} I(u) \to +\infty$ *as* $k \to \infty$,
(C_3) $\max_{u \in Y_k, \|u\|=\rho_k} I(u) \leqslant 0$,
(C_4) I *satisfies* $(\mathrm{PS})_c$ *condition for every* $c > 0$,

then I *has a sequence of critical values tending to* $+\infty$.

Proposition 3.2 *(Dual fountain theorem). (See [3,20].) Assume* (C_1) *is satisfied and there is a* $k_0 > 0$ *such that for each* $k \geqslant k_0$, *there exists* $\rho_k > r_k > 0$ *such that*

(D_1) $\inf_{u \in Z_k, \|u\|=\rho_k} I(u) \geqslant 0$,
(D_2) $b_k := \max_{u \in Y_k, \|u\|=r_k} I(u) \leqslant 0$,
(D_3) $d_k := \inf_{u \in Z_k, \|u\| \leqslant \rho_k} I(u) \to 0$ *as* $k \to \infty$,
(D_4) I *satisfies* $(\mathrm{PS})_c^*$ *condition for every* $c \in [d_{k_0}, 0)$,

then I *has a sequence of negative critical values converging to* 0.

Remark 3.2. I satisfies the $(\mathrm{PS})_c^*$ condition means that: if any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to \infty$, $u_{n_j} \in Y_{n_j}$, $I(u_{n_j}) \to c$ and $(I|_{Y_{n_j}})'(u_{n_j}) \to 0$, then $\{u_{n_j}\}$ contains a subsequence converging to critical point of I. It is obvious that if I satisfies the $(\mathrm{PS})_c^*$ condition, then I satisfies the $(\mathrm{PS})_c$ condition.

Proof of Theorem 3.3. Firstly, we show that I satisfies the $(\mathrm{PS})_c^*$ condition for every $c \in R$. Suppose $\{u_{n_j}\} \subset X$, $n_j \to \infty$, $u_{n_j} \in Y_{n_j}$, $I(u_{n_j}) \to c$ and $(I|_{Y_{n_j}})'(u_{n_j}) \to 0$. Similar to the process of verifying the (PS) condition in [6], we can obtain the boundedness of $\{\|u_{n_j}\|\}$. Going, if necessary, to a subsequence, we can assume that $u_{n_j} \rightharpoonup u$ in X. As $X = \overline{\bigcup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence

$$\begin{aligned}
&\lim_{n_j \to \infty} I'(u_{n_j})(u_{n_j} - u) \\
&\quad = \lim_{n_j \to \infty} I'(u_{n_j})(u_{n_j} - v_{n_j}) + \lim_{n_j \to \infty} I'(u_{n_j})(v_{n_j} - u) \\
&\quad = \lim_{n_j \to \infty} (I|_{Y_{n_j}})'(u_{n_j})(u_{n_j} - v_{n_j}) = 0.
\end{aligned} \tag{3.20}$$

Since I' is of (S_+) type, we can obtain $u_{n_j} \to u$, furthermore we have $I'(u_{n_j}) \to I'(u)$.

Next, let us show that $I'(u) = 0$. Taking arbitrary $w_k \in Y_k$, notice that when $n_j \geqslant k$ we have

$$\begin{aligned}
I'(u)w_k &= \big(I'(u) - I'(u_{n_j})\big)w_k + I'(u_{n_j})w_k \\
&= \big(I'(u) - I'(u_{n_j})\big)w_k + (I|_{Y_{n_j}})'(u_{n_j})w_k.
\end{aligned} \tag{3.21}$$

By going to limit in the right side of above equations, we have

$$I'(u)w_k = 0, \quad \forall w_k \in Y_k, \tag{3.22}$$

so $I'(u) = 0$. This shows that I satisfies the $(\mathrm{PS})_c^*$ condition for every $c \in R$. By Remark 3.2, I also satisfies the $(\mathrm{PS})_c$ condition for every $c \in R$.

Proof of conclusion (1). Let us verify for I the conditions in the fountain theorem item by item.

(C_2) For $k = 1, 2, \dots$, write

$$\theta_k = \sup_{v \in Z_k, \|v\| \leqslant 1} \int_\Omega \frac{\lambda}{m} |v|^m \, dx, \qquad \beta_k = \sup_{v \in Z_k, \|v\| \leqslant 1} \int_\Omega \frac{\mu}{n} |v|^m \, dx, \tag{3.23}$$

then $\theta_k, \beta_k > 0$ and $\theta_k, \beta_k \to 0$ as $k \to \infty$.

When $u \in Z_k$ and $\|u\| \geqslant 1$

$$I(u) \geqslant \|u\|^{p^-} - \|u\|^m \theta_k - \|u\|^n \beta_k. \tag{3.24}$$

For sufficiently large k, we have $\theta_k < \frac{1}{2}$. Since $m < p^-$, we get

$$I(u) \geqslant \frac{1}{2} \|u\|^{p^-} - \|u\|^n \beta_k. \tag{3.25}$$

Let

$$r_k = \left(\frac{p^-}{2n\beta_k}\right)^{\frac{1}{n-p^-}}, \tag{3.26}$$

when $u \in Z_k$ and $\|u\| = r_k$, for sufficiently large k,

$$I(u) \geqslant \left(\frac{1}{2} - \frac{p^-}{2n}\right)\left(\frac{1}{\beta_k}\right)^{\frac{p^-}{n-p^-}}\left(\frac{p^-}{2n}\right)^{\frac{p^-}{n-p^-}}. \tag{3.27}$$

Now $\beta_k \to 0$ implies

$$\inf_{u\in Z_k,\, \|u\|=r_k} I(u) \to +\infty \quad \text{as } k \to \infty. \tag{3.28}$$

So (C_2) is satisfied.

(C_3) For $k = 1, 2, \ldots,$ set

$$e_k = \inf_{v\in Y_k,\, \|v\|=1} \int_\Omega \frac{\mu}{n}|v|^n\,dx, \tag{3.29}$$

then $e_k > 0$. For any $v \in Y_k$ with $\|v\| = 1$ and $t > 1$, by Lemma 2.3, we have

$$I(tv) \leqslant t^{p^+} - e_k t^n. \tag{3.30}$$

As $n > p^+$, there exists $\rho_k > r_k$ such that $t = \rho_k$ concludes $I(tv) \leqslant 0$, and then

$$\max_{u\in Y_k,\, \|u\|=\rho_k} I(u) \leqslant 0. \tag{3.31}$$

Hence (C_3) is satisfied.

Conclusion (1) is reached by the fountain theorem.

Proof of conclusion (2). Let us show that I satisfies conditions in the dual fountain theorem item by item.

(D_1) Let θ_k and β_k be defined by (3.23), when $v \in Z_k$, $\|v\| = 1$ and $0 < t < 1$ we have

$$I(tv) \geqslant t^{p^+} - \theta_k t^m - \beta_k t^n \geqslant t^{p^+} - \theta_k t^m - \beta_k t^{p^+}. \tag{3.32}$$

For sufficiently large k we have $\beta_k < \frac{1}{2}$, thus

$$I(tv) \geqslant \frac{1}{2}t^{p^+} - \theta_k t^m. \tag{3.33}$$

Taking $\rho_k = (2\theta_k)^{\frac{1}{p^+-m}}$, then for sufficiently large k, $\rho_k < 1$. When $t = \rho_k$, $v \in Z_k$ with $\|v\| = 1$, we have $I(tv) \geqslant 0$. So for sufficiently large k,

$$\inf_{u\in Z_k,\, \|u\|=\rho_k} I(u) \geqslant 0.$$

Hence (D_1) is satisfied.

(D_2) Let

$$\delta_k = \inf_{v\in Y_k,\, \|v\|=1} \int_\Omega \frac{\lambda}{m}|v|^m\,dx, \tag{3.34}$$

then $\delta_k > 0$. For $v \in Y_k$, $\|v\| = 1$ and $0 < t < 1$,

$$I(tv) \leqslant t^{p^-} - \delta_k t^m. \tag{3.35}$$

Since $m < p^-$, there exists an $r_k \in (0, \rho_k)$ such that when $r = r_k$, $I(tv) < 0$. So we obtain

$$b_k := \max_{u\in Y_k,\, \|u\|=r_k} I(u) < 0.$$

Hence (D_2) is satisfied.

(D_3) Since $Y_k \cap Z_k \neq \emptyset$ and $r_k < \rho_k$, we have

$$d_k := \inf_{u\in Z_k,\, \|u\|\leqslant\rho_k} I(u) \leqslant b_k := \max_{u\in Y_k,\, \|u\|=r_k} I(u) < 0. \tag{3.36}$$

Using (3.33), for $v \in Z_k$, $\|v\| = 1$, $0 \leqslant t \leqslant \rho_k$ and $u = tv$

$$I(u) = I(tv) \geqslant \frac{1}{2}t^{p^-} - \theta_k t^m \geqslant -\theta_k t^m \geqslant -\theta_k(\rho_k)^m \geqslant -\theta_k. \tag{3.37}$$

Hence $d_k \to 0$, i.e., (D_3) is satisfied.

Conclusion (2) is reached by the dual fountain theorem. □

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